Ultracoherence and Canonical Transformations

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The (in)finite dimensional symplectic group of homogeneous canonical transformations is represented on the bosonic Fock space by the action of the group on the ultracoherent vectors, which are generalizations of the coherent states.

1 Introduction

The linear canonical transformations are an important tool to study the structure and the dynamics of quantum systems. An incomplete list of the literature on this subject is [1, 5, 6, 7, 13, 17, 22, 24, 26]. The aim of this paper is to give a self-contained presentation of canonical transformations in quantum mechanics and in Fock space quantum field theory using ultracoherent vectors. These vectors are generated by the group of all linear canonical transformations acting on the vacuum; they are Gaussian pure states and include the well known coherent vectors and the squeezed vacua of quantum optics. The name *ultracoherence* is taken from [29], where canonical transformations are investigated with an algebra of normal ordered operators.

There is an extensive literature about the representations of the finite dimension symplectic group. The special role of exponential vectors is already emphasized in the publications of Bargmann [5] and Itzykson [13] using the complex wave representation (or Bargmann-Segal-Fock representation) of the Fock space. These authors use the reproducing kernel property of the complex wave representation and construct the kernel functions for the operators of the representation, see also [16]. Our basic ansatz in Sect. 5.1.1 is motivated by formulas in [5, 13, 16]. But the spirit of our construction is closer to [28]. These authors investigate the action of the symplectic group on Gaussian pure states in the Schrödinger representation of quantum mechanics. Transferred to the Fock space language, the Gaussian pure states are the ultracoherent states for the quantum mechanics of a finite number of degrees of freedom.

The representation of the infinite dimensional symplectic group on the bosonic Fock space have been studied in [6] using a symbolic calculus related to the complex wave representation and in [24] with rigorous normal ordering expansions. The aim of our paper is to construct the representation on a minimal set of vectors in the Fock space, which is stable against Weyl transformations and homogeneous canonical transformations. The linear span of these vectors, the ultracoherent vectors, provide a natural domain for the (possibly unbounded) generators of one-parameter subgroups of the infinite dimensional symplectic group.

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The plan of the paper is as follows. In Section 2 we briefly discuss Hilbert and bosonic Fock spaces. We define and discuss the exponential vectors – related to coherent states – and the more general ultracoherent vectors. In Section 3 we introduce the algebra of Weyl operators, which defines the canonical structure with bounded operators. In Section 4 we investigate the symplectic group on finite and infinite dimensional Hilbert spaces. In the case of infinite dimensions the symplectic transformations can be implemented by unitary operators on the Fock space only if an additional Hilbert-Schmidt condition is satisfied [7, 26]. Under this restriction we construct in Section 5.1 a unitary ray representation of the symplectic group on the bosonic Fock space by defining the action of this group on exponential and ultracoherent vectors. The intertwining relations of this representation with the algebra of Weyl operators are given in Section 5.2. In the concluding Section 6 we indicate possible applications of our approach. Some detailed calculations for ultracoherent vectors are given in the Appendix A.

2 Fock space and ultracoherent vectors

2.1 Hilbert spaces and Fock spaces

In this section we recapitulate some basic notations about Hilbert spaces and Fock spaces of symmetric tensors. Let $\mathcal{H}$ be a complex separable Hilbert space with inner product $(f \mid g)$ and with an antiunitary involution $f \to f^*$, $f^{**} \equiv (f^*)^* = f$. Then the mapping

$$f,\, g \to \langle f \mid g \rangle := (f^* \mid g) \in \mathbb{C} \tag{1}$$

is a symmetric bilinear form $\langle f \mid g \rangle = \langle g \mid f \rangle$. The underlying real Hilbert space of $\mathcal{H}$ is denoted as $\mathcal{H}_{\mathbb{R}}$. This space has the inner product $(f \mid g)_{\mathbb{R}} = \mathrm{Re}\ (f \mid g)$
$= \frac{1}{2}\left((f \mid g) + (f^* \mid g^*)\right)$. As a point set the spaces $\mathcal{H}$ and $\mathcal{H}_{\mathbb{R}}$ coincide. For some calculations it is advantageous to identify $\mathcal{H}_{\mathbb{R}}$ with diagonal subspace $\mathcal{H}_{diag}$ of $\mathcal{H} \times \mathcal{H}^*$. This space is defined as the set of all elements $\begin{pmatrix} f \\ g^* \end{pmatrix} \in \mathcal{H} \times \mathcal{H}^*$ which satisfy $g = f$.

We use the following notations for linear operators: the space of all bounded operators A with operator norm $\|A\|$ is $\mathcal{L}(\mathcal{H})$, the space of all Hilbert-Schmidt operators A with norm $\|A\|_{HS} = \sqrt{\mathrm{tr}_{\mathcal{H}} A^+ A}$ is $\mathcal{L}_2(\mathcal{H})$, the space of all trace class or nuclear operators A with norm $\|A\|_1 = \mathrm{tr}_{\mathcal{H}} \sqrt{A^+ A}$ is $\mathcal{L}_1(\mathcal{H})$. For operators $A \in \mathcal{L}(\mathcal{H})$ the adjoint operator is denoted by A^+. The complex conjugate operator $\bar{A}$ and the transposed operator A^T are defined by the identities

$$\bar{A}f = (Af^*)^*\,,\ A^T f = \left(A^+ f^*\right)^* \tag{2}$$

for all $f \in \mathcal{H}$. The usual relations $A^+ = \left(\bar{A}\right)^T = \overline{(A^T)}$ are valid. An operator A with the property $A = A^T$ is called a *transposition-symmetric operator*. It satisfies the symmetry relation $\langle f \mid Ag \rangle = \langle Af \mid g \rangle$ for all $f, g \in \mathcal{H}$.

Let $\mathcal{H}^{\otimes n}$, $n \in \mathbb{N}$, be the algebraic n-th tensor power of the Hilbert space $\mathcal{H}$. The norm of $\mathcal{H}^{\otimes n}$ is fixed with the normalization $\|f_1 \otimes f_2 \otimes ... \otimes f_n\|_n = \sqrt{n!} \prod_{j=1}^n \|f_j\|$ for the product of n vectors $f_j \in \mathcal{H}$. The completion of $\mathcal{H}^{\otimes n}$, $n \geq 2$, with this norm is the Hilbert space $\widehat{\mathcal{H}^{\otimes n}}$. The projection operator $P_n : \widehat{\mathcal{H}^{\otimes n}} \to \widehat{\mathcal{H}^{\otimes n}}$ onto symmetric tensors of

degree n is uniquely given by the prescription $P_n(f_1 \otimes f_2 \otimes ... \otimes f_n) = (n!)^{-1} \sum_\sigma f_{\sigma(1)} \otimes f_{\sigma(2)} \otimes ... \otimes f_{\sigma(n)}$ where σ runs over all permutations of the numbers $\{1, ..., n\}$. The algebraic space of symmetric tensors is $\mathcal{H}^{\vee n} = P_n \mathcal{H}^{\otimes n}$ and the completion is the Hilbert space $\widehat{\mathcal{H}^{\vee n}} = P_n \widehat{\mathcal{H}^{\otimes n}}$. With $\widehat{\mathcal{H}^{\otimes 0}} = \mathbb{C}$ and $\widehat{\mathcal{H}^{\otimes 1}} = \mathcal{H}^{\otimes 1} = \mathcal{H}$ the Fock space of all tensors is the direct sum $\mathcal{T}(\mathcal{H}) = \oplus_{n=0}^{\infty} \widehat{\mathcal{H}^{\otimes n}}$ where the norm is defined by $\|F\|^2 = \sum_{n=0}^{\infty} \|F_n\|_n^2$ if $F = \sum_{n=0}^{\infty} F_n$, $F_n \in \widehat{\mathcal{H}^{\otimes n}}$. With $\widehat{\mathcal{H}^{\vee 0}} = \mathbb{C}$ and $\widehat{\mathcal{H}^{\vee 1}} = \mathcal{H}^{\vee 1} = \mathcal{H}$ the Hilbert sum $\mathcal{S}(\mathcal{H}) = \oplus_{n=0}^{\infty} \widehat{\mathcal{H}^{\vee n}}$ defines the Fock space of symmetric tensors as subspace of $\mathcal{T}(\mathcal{H})$. The symmetric tensor product of the tensors $F \in \widehat{\mathcal{H}^{\vee m}}$ and $G \in \widehat{\mathcal{H}^{\vee n}}$ is defined as $F \vee G = P_{m+n}(F \otimes G)$. Then the usual normalization $\|f_1 \vee ... \vee f_n\|_n^2 = \mathrm{per}\,((f_k \mid f_l))$ with the permanent follows for the norm of the product of n vectors $f_j \in \mathcal{H}$, $j = 1, ..., n$.

Since $\|F \otimes G\|_{m+n} \leq \sqrt{\frac{(m+n)!}{m!n!}} \|F\|_m \|G\|_n$ with the norm introduced above, we also have

$$\|F \vee G\|_{m+n} \leq \sqrt{\frac{(m+n)!}{m!n!}} \|F\|_m \|G\|_n . \tag{3}$$

By linear extension the symmetric tensor product is extended to the algebraic sum $\mathcal{S}_{fin}(\mathcal{H}) = \oplus_n \widehat{\mathcal{H}^{\vee n}}$ (linear subset of all $F \in \mathcal{S}(\mathcal{H})$ which have components in a finite number of subspaces $\widehat{\mathcal{H}^{\vee n}}$ only). The space $\mathcal{S}_{fin}(\mathcal{H})$ is an algebra with respect to the symmetric tensor product; the unit is the normalized basis vector $\mathbf{1}_{vac}$ (vacuum) of the space $\widehat{\mathcal{H}^{\vee 0}} = \mathbb{C}$. If we restrict the Hilbert spaces $\widehat{\mathcal{H}^{\vee n}}$ to the algebraic tensor spaces $\mathcal{H}^{\vee n}$ we obtain the algebra $\mathcal{S}^0_{alg}(\mathcal{H}) = \oplus_n \mathcal{H}^{\vee n}$, which is strictly smaller than $\mathcal{S}_{fin}(\mathcal{H})$ if $\dim \mathcal{H} = \infty$, but still dense in $\mathcal{S}(\mathcal{H})$.

The inner product of two elements F, G of $\mathcal{S}(\mathcal{H})$ is written as $(F \mid G)$. The antiunitary involution $f \to f^*$ on $\mathcal{H}$ can be uniquely extended to an antiunitary involution $F \to F^*$on $\mathcal{S}(\mathcal{H})$ with the rule $(F \vee G)^* = G^* \vee F^* = F^* \vee G^*$. The mapping $F, G \in \mathcal{S}(\mathcal{H}) \to \langle F \mid G \rangle := (F^* \mid G) \in \mathbb{C}$ is again a $\mathbb{C}-$bilinear symmetric form.

The normalizations of the symmetric tensor product used in this paper agree with those of [21]. The algebra $\Gamma_0 \mathcal{H}$ of [21] coincides with the algebra $\mathcal{S}^0_{alg}(\mathcal{H})$ defined above.

2.2 Exponential vectors

For all vectors $f \in \mathcal{H}$ the exponential series $\exp f = 1_{vac} + f + \frac{1}{2!} f \vee f + ...$ is absolutely summable within $\mathcal{S}(\mathcal{H})$ and it satisfies the usual factorization property $\exp f \vee \exp g = \exp(f + g)$, see [21] and also Appendix A.1 of this paper. The mapping $f \to \exp f$ is an entire analytic function[3]. The inner product of two exponential vectors is

$$(\exp f \mid \exp g) = \exp\,(f \mid g) . \tag{4}$$

Coherent states are the normalized exponential vectors $\exp\left(f - \frac{1}{2}\|f\|^2\right) \in \mathcal{S}(\mathcal{H})$. The linear span of all exponential vectors $\{\exp f \mid f \in \mathcal{H}\}$ will be denoted by $\mathcal{S}_{coh}(\mathcal{H})$. The involution of an exponential vector is $(\exp f)^* = \exp f^*$, $f \in \mathcal{H}$. Due to the factorization property $\exp f \vee \exp g = \exp(f + g)$ the set $\mathcal{S}_{coh}(\mathcal{H})$ is an algebra.

Lemma 1 *The set $\{\exp f \mid f \in \mathcal{H}\}$ of all exponential vectors is linearly independent and the linear span $\mathcal{S}_{coh}(\mathcal{H})$ of these vectors is dense in $\mathcal{S}(\mathcal{H})$.*

[3]In this article analyticity means the existence of norm convergent power series expansions as used e.g. in [11].

Proof. A proof is given in [9] § 2.1 and in [23] Proposition 19.4. □

A (bounded) operator on $\mathcal{S}(\mathcal{H})$ it is therefore uniquely determined, if this operator is known on all exponential vectors. Actually it is sufficient to define the operator on a set $\{\exp f \mid f \in \mathcal{D}\}$ where $\mathcal{D}$ is dense in $\mathcal{H}$, see Corollary 19.5 of [23]. As example a bounded operator $B \in \mathcal{L}(\mathcal{H})$ can be lifted to an operator $\Gamma(B)$ on $\mathcal{S}(\mathcal{H})$ using the prescription

$$\Gamma(B) \exp f := \exp Bf, \;\; f \in \mathcal{H}. \tag{5}$$

The operator $\Gamma(B)$ is continuous, if B is a contraction, i. e. $\|B\| \leq 1$; and it is isometric/unitary, if B is isometric/unitary. The statement about isometry is an immediate consequence of

Lemma 2 *Let T_0 be a linear operator $T_0 : \mathcal{S}_{coh}(\mathcal{H}) \to \mathcal{S}(\mathcal{H})$ which satisfies*

$$(T_0 \exp f \mid T_0 \exp g) = (\exp f \mid \exp g) = \exp{(f \mid g)} \tag{6}$$

for all $f, g \in \mathcal{H}$, then T_0 can be uniquely extended to a linear isometric mapping T on $\mathcal{S}(\mathcal{H})$.

Proof. The proof follows from Lemma 1 and from the Proposition 7.2 of [23]. □

We have already stated that the dense linear subsets $\mathcal{S}_{fin}(\mathcal{H})$ and $\mathcal{S}_{coh}(\mathcal{H})$ are algebras with respect to the symmetric tensor product. Take $F \in \mathcal{S}_{fin}(\mathcal{H})$ and $G \in \mathcal{S}_{coh}(\mathcal{H})$ then the product $F \vee G = G \vee F$ is defined, and the linear span of these products $\mathcal{S}_{alg}(\mathcal{H}) = span\ \{F \vee \exp g \mid F \in \mathcal{S}_{fin}(\mathcal{H}),\ g \in \mathcal{H}\}$ is again an algebra. The proof of the corresponding statement has been given for the algebra $\mathcal{S}^0_{alg}(\mathcal{H}) \subset \mathcal{S}_{alg}(\mathcal{H})$ generated by $\mathcal{S}^0_{fin}(\mathcal{H})$ and $\mathcal{S}_{coh}(\mathcal{H})$ in [21] Theorem 3.4. The essential estimate for this proof is (3), which applies to $\mathcal{H}^{\vee n}$ and to $\widehat{\mathcal{H}^{\vee n}}$. The proof of [21] can therefore easily be extended to the algebra $\mathcal{S}_{alg}(\mathcal{H})$.

For all $f,\ g \in \mathcal{H}$ the product $f \vee \exp g$ is an element of $\mathcal{S}(\mathcal{H})$. Given a vector $f \in \mathcal{H}$ the creation operator $a^+(f)$ and the corresponding annihilation operator $a(f)$ are uniquely determined by

$$\begin{aligned} a^+(f) \exp g &= f \vee \exp g \\ a(f) \exp g &= \langle f \mid g \rangle \exp g. \end{aligned} \tag{7}$$

These operators are related by $(a^+(f))^+ = a(f^*)$, and they satisfy the canonical commutation relations $[a(f), a^+(g)] = \langle f \mid g \rangle\ I,\;\; [a^+(f), a^+(g)] = [a(f), a(g)] = 0$ for all $f,\ g \in \mathcal{H}$. These relations are equivalent to

$$\left[a^+(f) - a(f^*), a^+(g) - a(g^*)\right] = -2i\omega(f,g)\, I \tag{8}$$

with

$$\omega(f,g) := \frac{1}{2i} \left(\langle f^* \mid g \rangle - \langle f \mid g^* \rangle\right) = \mathrm{Im}\,(f \mid g) \in \mathbb{R}, \tag{9}$$

which is an $\mathbb{R}$-bilinear continuous skew symmetric form on the Hilbert space $\mathcal{H}$ or, more precisely, on the underlying real space $\mathcal{H}_{\mathbb{R}}$. The creation and annihilation operators are unbounded operators. A domain of definition, on which these operators and their commutators are meaningful is the algebra $\mathcal{S}_{alg}(\mathcal{H})$.

2.3 Ultracoherent vectors

The space $\mathcal{L}_2(\mathcal{H})$ of Hilbert-Schmidt operators on $\mathcal{H}$ is a Hilbert space with the Hilbert-Schmidt norm $\|A\|_{HS} = \sqrt{\mathrm{tr}A^+A}$. The restriction to transposition-symmetric Hilbert-Schmidt operators $\left\{A \in \mathcal{L}_2(\mathcal{H}) \mid A = A^T\right\}$ is a closed subspace of $\mathcal{L}_2(\mathcal{H})$. This space is called $\mathcal{L}_{2sym}(\mathcal{H})$. There exists a linear isomorphism between $\mathcal{L}_{2sym}(\mathcal{H})$ and the Hilbert space $\widehat{\mathcal{H}^{\vee 2}}$ of tensors of second degree:

Lemma 3 *Let A be an operator in $\mathcal{L}_{2sym}(\mathcal{H})$, then there exists a unique tensor of second degree, in the sequel denoted by $\Omega(A)$, such that*

$$\langle \Omega(A) \mid f \vee g\rangle = \langle f \mid Ag\rangle = \langle g \mid Af\rangle \tag{10}$$

for all f, $g \in \mathcal{H}$. The mapping $A \in \mathcal{L}_{2sym}(\mathcal{H}) \to \Omega(A) \in \widehat{\mathcal{H}^{\vee 2}}$ is an invertible continuous linear transformation between the spaces $\mathcal{L}_{2sym}(\mathcal{H})$ and $\widehat{\mathcal{H}^{\vee 2}}$. The respective norms are related by $\|\Omega(A)\|_2^2 = \frac{1}{2}\|A\|_{HS}^2$.

Proof. As $\langle \Omega(A) \mid f \vee g\rangle = 0$ for all f, $g \in \mathcal{H}$ implies $\Omega(A) = 0$, the tensor $\Omega(A)$ is uniquely determined by (10). For the construction of $\Omega(A)$ we choose a real orthonormal basis $e_\mu = e_\mu^*$ of the Hilbert space $\mathcal{H}$. Then any transposition-symmetric Hilbert-Schmidt operator A has matrix elements $A_{\mu\nu} = (e_\mu \mid A\, e_\nu) = \langle e_\mu \mid A\, e_\nu\rangle = A_{\nu\mu}$ which are square summable $\sum_{\mu\nu} |A_{\mu\nu}|^2 = \|A\|_{HS}^2 < \infty$. The tensor $\Omega(A) := \frac{1}{2}\sum_{\mu\nu} A_{\mu\nu}\, e_\mu \vee e_\nu \in \widehat{\mathcal{H}^{\vee 2}}$ then satisfies the identity (10). The tensor norm of $\Omega(A)$ is calculated as $\frac{1}{2}\sum_{\mu\nu} |A_{\mu\nu}|^2$ and the norm identity follows.

On the other hand, given a tensor $F \in \widehat{\mathcal{H}^{\vee 2}}$ we have $F = \frac{1}{2}\sum_{\mu\nu} F_{\mu\nu}\, e_\mu \vee e_\nu$ with coefficients $F_{\mu\nu} = F_{\upsilon\mu} \in \mathbb{C}$, which are square summable. Then the operator $f \in \mathcal{H} \to Af = \sum_\mu F_{\mu\nu}\langle e_\mu \mid f\rangle\, e_\upsilon \in \mathcal{H}$ is obviously a transposition-symmetric Hilbert-Schmidt operator with $\Omega(A) = F$. □

Definition 1 *The set of all Hilbert–Schmidt operators $A \in \mathcal{L}_{2sym}(\mathcal{H})$ with an operator norm strictly less than one, $\|A\| < 1$, is called the Siegel (unit) disc. It is denoted by $\mathbf{D}_1$.*

For Hilbert spaces with finite dimensions this disc has been introduced by Siegel [27]. The set $\mathbf{D}_1$ is open and convex, and it is stable against transformations $A \to UAU^T$ with a unitary operator U. The last statement follows from $\mathrm{tr}_{\mathcal{H}}\bar{U}A^+U^+UAU^T = \mathrm{tr}_{\mathcal{H}}A^+A$ and $\left\|UAU^T\right\| \le \|A\|$.

In [17] it has been derived that the exponential series

$$\exp\Omega(A) = 1_{vac} + \Omega(A) + \frac{1}{2!}\Omega(A) \vee \Omega(A) + ... \tag{11}$$

converges within the Fock space $\mathcal{S}(\mathcal{H})$ if $A \in \mathbf{D}_1$. The convergence is uniform for each subset $\mathbf{D}_1^{c,\delta} = \{A \in \mathbf{D}_1 \mid \|A\|_{HS} \le c < \infty,\ \|A\| \le \delta < 1\}$. The mapping $A \in \mathbf{D}_1 \to \exp\Omega(A) \in \mathcal{S}(\mathcal{H})$ is analytic. The inner product of two of these exponentials can be calculated as

$$(\exp\Omega(A) \mid \exp\Omega(B)) = \left(\det{}_{\mathcal{H}}\left(I - A^+B\right)\right)^{-\frac{1}{2}} = \left(\det{}_{\mathcal{H}}\left(I - BA^+\right)\right)^{-\frac{1}{2}}. \tag{12}$$

The proof of this identity follows from Theorem 2 of [17], but see also Appendix A.3. The identity (12) implies that $\exp \Omega(A) \in \mathcal{S}(\mathcal{H})$ if and only if $A \in \mathbf{D}_1$.

In Appendix A.1 we prove that the symmetric tensor product of the tensors $\exp \Omega(A)$, $A \in \mathbf{D}_1$, and $\exp f$, $f \in \mathcal{H}$, is defined within the Fock space $\mathcal{S}(\mathcal{H})$. For any operator $A \in \mathbf{D}_1$ and for any $f \in \mathcal{H}$ we now define the *ultracoherent vector*

$$\Phi(A, f) := \exp \Omega(A) \vee \exp f = \exp f \vee \exp \Omega(A) \in \mathcal{S}(\mathcal{H}). \tag{13}$$

The result (12) can be extended to the inner product of two ultracoherent vectors

$$\begin{aligned} &(\Phi(A, f) \mid \Phi(B, g)) \\ &= (\det{}_{\mathcal{H}} (I - A^+B))^{-\frac{1}{2}} \exp\left(\tfrac{1}{2} \langle f^* \mid C f^* \rangle + \langle f^* \mid (I - BA^+)^{-1} g \rangle + \tfrac{1}{2} \langle g \mid D\, g \rangle\right) \end{aligned} \tag{14}$$

with

$$\begin{aligned} C = B(I - A^+B)^{-1} = (I - BA^+)^{-1} B \\ D = A^+(I - BA^+)^{-1} = (I - A^+B)^{-1} A^+. \end{aligned} \tag{15}$$

Proofs of this identity are given in the Appendices A.2 and A.3.

Remark 1 *For $F \in \mathcal{S}(\mathcal{H})$ the function*

$$\Phi_F(z^*) = (\exp z \mid F) = \langle \exp z^* \mid F \rangle \tag{16}$$

is entire antianalytic in the variable $z \in \mathcal{H}$, and the tensor F is uniquely determined by this function. We denote the linear space of all functions $\{\Phi_F(z^*) \mid F \in \mathcal{S}(\mathcal{H})\}$ by $\mathcal{B}$. Then $\mathcal{B}$ can be equipped with the Hilbert space topology induced by the topology of $\mathcal{S}(\mathcal{H})$, i.e., the inner product $(\Phi_F \parallel \Phi_G)$ of two functions Φ_F and Φ_G is defined as $(\Phi_F \parallel \Phi_G) := (F \mid G)$. With this structure the space $\mathcal{B}$ becomes a Hilbert space with the reproducing kernel $\exp \langle z^* \mid w \rangle$. *This representation of the bosonic Fock space is called the complex wave representation or Bargmann-Fock representation, see [4, 25] and the more recent publications [2, 10, 21]. The exponential vectors and the ultracoherent vectors have a simple representation in this space:* $\exp f \in \mathcal{S}(\mathcal{H})$ *corresponds to the exponential function* $(\exp z \mid \exp f) = \exp \langle z^* \mid f \rangle$ *and the ultracoherent vectors are given by (72).*
If $F \in \mathcal{S}_{coh}(\mathcal{H})$ then $\Phi_F(z^*)$ is a tame function, i.e., it depends only on a finite number of variables $\langle z^* \mid f_j \rangle$, $f_j \in \mathcal{H}$, $j = 1, ..., N$. Let $\upsilon(dz, dz^*)$ be the canonical Gaussian promeasure on the Hilbert space $\mathcal{H}_\mathbb{R}$, then tame functions can be integrated, and the identity

$$\int_{\mathcal{H}_\mathbb{R}} \overline{\Phi_F(z^*)} \Phi_G(z^*) \upsilon(dz, dz^*) = (F \mid G) \tag{17}$$

holds for all $F, G \in \mathcal{S}_{coh}(\mathcal{H})$. If $A \in \mathcal{D}_1$ is a finite rank operator, then (72) is a tame function. The integral (17) can be used to calculate the inner product (14) – first for finite rank operators A and B and then by a continuity argument for general $A, B \in \mathcal{D}_1$. A calculation of the integral for finite dimensional Hilbert spaces can be found in Appendix II of [13]. The proof of (14), which we present in Appendix A, does not use this technique.

3 Weyl operators and canonical transformations

3.1 Weyl operators

In this Section we recapitulate some properties of Weyl operators needed for the subsequent investigations. The Weyl operators $W(h)$, $h \in \mathcal{H}$, are defined on the linear span of the exponential vectors by

$$W(h) \exp f = \mathrm{e}^{-(h|f)-\frac{1}{2}\|h\|^2} \exp(f+h) = \mathrm{e}^{-\langle h^*|f\rangle - \frac{1}{2}\langle h^*|h\rangle} \exp(f+h). \tag{18}$$

It is straightforward to derive the identity

$$(W(h) \exp f \mid W(h) \exp g) = \mathrm{e}^{(f|g)} = (\exp f \mid \exp g)$$

for $f, g, h \in \mathcal{H}$. Then Lemma 2 implies that $W(h)$ is isometric. On the other hand we have $W(h)W(-h) = id$ and $W(h)$ is invertible with $W^{-1}(h) = W(-h)$. The Weyl operators can therefore be extended to unitary operators on the Fock space $\mathcal{S}(\mathcal{H})$. Calculating $W(f)W(g) \exp h$ and $W(f+g) \exp h$ we obtain the *Weyl relations*

$$W(f)W(g) = \mathrm{e}^{-i\omega(f,g)} W(f+g) \tag{19}$$

with the skew symmetric form (9) on the underlying real space $\mathcal{H}_{\mathbb{R}}$. The identity (19) defines the canonical structure on the Fock space $\mathcal{S}(\mathcal{H})$. The Weyl relations are equivalent to the canonical commutation relations (8). The advantage of the Weyl relations is that they are formulated with bounded operators.

The action of the Weyl operator on the ultracoherent vector is calculated in Appendix A.2 as

$$W(h)\Phi(A, f) = \mathrm{e}^{-\frac{1}{2}\|h\|^2 + \frac{1}{2}\langle h^*|Ah^* - 2f\rangle} \Phi(A, f + h - Ah^*). \tag{20}$$

As well known, the Weyl operators have a simple representation in terms of the creation and annihilation operators. Differentiating $W(\lambda h) \exp f$ with respect to $\lambda \in \mathbb{R}$ and comparing the result with (7) we obtain the usual representation of the Weyl operator $W(h) = \exp\left(a^+(h) - a(h^*)\right)$. Using (18) together with (7) we get the relations

$$\begin{aligned} W(h)a^+(f)W^+(h) &= a^+(f) - (h \mid f) = a^+(f) - \langle f \mid h^* \rangle \,, \\ W(h)a(f)W^+(h) &= a(f) - (f^* \mid h) = a(f) - \langle f \mid h \rangle \,. \end{aligned} \tag{21}$$

3.2 Canonical transformations

Canonical transformations are unitary operators S on $\mathcal{S}(\mathcal{H})$ which preserve the canonical commutation relations (8). To avoid any discussion about the domain of the operators $Sa^+(f)S^+$ and $Sa(f^*)S^+$ we demand the invariance of the Weyl relations (19)

$$SW(f)S^+SW(g)S^+ = SW(f)W(g)S^+ = \mathrm{e}^{-i\mathrm{Im}(f|g)} SW(f+g)S^+. \tag{22}$$

There are two types of linear canonical transformations:

1. The inhomogeneous Transformations generate a c-number shift for the creation and annihilation operators. These transformations are given by the unitary Weyl operators, as can be seen from the relations (21). The invariance of the Weyl relations

   $$W(h)W(f)W(g)W(-h) = \mathrm{e}^{-i\mathrm{Im}(f|g)} W(h)W(f+g)W(-h)$$

   with $S = W(h)$, $h \in \mathcal{H}$, as canonical transformation easily follow from (19).

2. The homogeneous canonical transformations generate linear transformations between the creation and annihilation operators

$$\begin{aligned} Sa^+(f)S^+ &= a^+(Uf) - a(\bar{V}f), \\ Sa(f)S^+ &= -a^+(Vf) + a(\bar{U}f). \end{aligned} \tag{23}$$

Here U and V are bounded linear transformations on $\mathcal{H}$. The relations (23) imply $S\left(a^+(f) - a(f^*)\right)S^+ = a^+(Uf + Vf^*) - a\left(\bar{V}f + \bar{U}f^*\right)$. The canonical transformations of this type are usually called *Bogoliubov transformations*. The Weyl form of these transformations is

$$SW(f)S^+ = W(Uf + Vf^*). \tag{24}$$

The Weyl relations (and consequently the canonical commutation relations) are preserved, if the skew symmetric form (9) is invariant against the $\mathbb{R}$-linear mapping

$$f \in \mathcal{H}_{\mathbb{R}} \to R(U,V)f := Uf + Vf^* \in \mathcal{H}_{\mathbb{R}}, \tag{25}$$

i.e.

$$\omega(Rf, Rg) \equiv \omega(Uf + Vf^*, Ug + Vg^*) = \omega(f, g). \tag{26}$$

for all $f, g \in \mathcal{H}_{\mathbb{R}}$.

The transformations (25) which satisfy (26) form the *symplectic group* of the Hilbert space $\mathcal{H}_{\mathbb{R}}$, and the transformations (24) generate a unitary ray representation of this group on the Fock space $\mathcal{S}(\mathcal{H})$. So far we have only assumed that U and V are bounded linear operators on $\mathcal{H}$ (and consequently also on $\mathcal{H}_{\mathbb{R}}$). For infinite dimensional Hilbert spaces $\mathcal{H}$ – needed for quantum field theory – an additional constraint turns out to be necessary: In order to obtain a unitary ray representation on $\mathcal{S}(\mathcal{H})$ the operator V has to be a Hilbert-Schmidt operator, see [7, 26].

4 The symplectic group

4.1 Definition

In this Section we give a more explicit definition of the symplectic transformations and recapitulate some identities which are needed for the subsequent calculations. We identify $\mathcal{H}_{\mathbb{R}}$ with the diagonal subspace $\mathcal{H}_{diag} \subset \mathcal{H} \times \mathcal{H}^*$, see the beginning of Sect. 2.1. The space $\mathcal{H} \times \mathcal{H}^*$ has elements $\begin{pmatrix} f \\ g^* \end{pmatrix}$ with $f, g \in \mathcal{H}$. On $\mathcal{H} \times \mathcal{H}^*$ we define the operators $\Delta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ and $\hat{M} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$. The matrix array $\hat{R} = \begin{pmatrix} U & V \\ X & Y \end{pmatrix}$ of operators $U, V, X, Y \in \mathcal{L}(\mathcal{H})$ yields a bounded linear operator on $\mathcal{H} \times \mathcal{H}^*$

$$\begin{pmatrix} U & V \\ X & Y \end{pmatrix} \begin{pmatrix} f \\ g^* \end{pmatrix} = \begin{pmatrix} Uf + Vg^* \\ Xf + Yg^* \end{pmatrix}.$$

Definition 2 *The operator $\hat{R}$ is a symplectic transformation, if it satisfies the constraints*

$$\hat{R}\hat{M}\hat{R}^{+} = \hat{M} \text{ and } \Delta\overline{\hat{R}}\Delta = \hat{R}. \tag{27}$$

The set of all these transformations is denoted by $\widehat{S}p(\mathcal{H})$.

The second constraint in (27) implies that $\hat{R}$ has the form

$$\hat{R} = \hat{R}(U,V) = \begin{pmatrix} U & V \\ \bar{V} & \bar{U} \end{pmatrix}. \tag{28}$$

Thereby $\bar{U}$ and $\bar{V}$ are the complex conjugate operators of U and V, respectively, as defined in Sect 2.1. The identity operator $\hat{I} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$ is an element of $\widehat{S}p(\mathcal{H})$. The product of two matrix operators (28)

$$\hat{R}_2\hat{R}_1 = \begin{pmatrix} U_2 & V_2 \\ \bar{V}_2 & \bar{U}_2 \end{pmatrix}\begin{pmatrix} U_1 & V_1 \\ \bar{V}_1 & \bar{U}_1 \end{pmatrix} = \begin{pmatrix} U_2U_1 + V_2\bar{V}_1 & U_2V_1 + V_2\bar{U}_1 \\ \bar{V}_2U_1 + \bar{U}_2\bar{V}_1 & \bar{V}_2V_1 + \bar{U}_2\bar{U}_1 \end{pmatrix} \tag{29}$$

is also an element of $\widehat{S}p(\mathcal{H})$. From the first identity of (27) follows the inverse of $\hat{R}$ as

$$\hat{R}^{-1} = \hat{M}\hat{R}^{+}\hat{M} = \begin{pmatrix} U^{+} & -V^{T} \\ -V^{+} & U^{T} \end{pmatrix}. \tag{30}$$

On the other hand, if a matrix operator (28) satisfies $\hat{R}^{-1} = \hat{M}\hat{R}^{+}\hat{M}$, then the conditions of Definition 2 apply to $\hat{R}$, and $\hat{R}$ is an element of $\widehat{S}p(\mathcal{H})$. From $\hat{R}^{-1}\hat{R} = I$ we have $\hat{R}^{+}\hat{M}\hat{R} = \hat{M}$ such that

$$\hat{R}^{+} = \begin{pmatrix} U^{+} & V^{T} \\ V^{+} & U^{T} \end{pmatrix} \in \widehat{S}p(\mathcal{H}). \tag{31}$$

But then also the operator (30) is an element of $\widehat{S}p(\mathcal{H})$, and the set $\widehat{S}p(\mathcal{H})$ is a group with identity $\hat{I}$ and multiplication (29).

The (equivalent) identities $\hat{R}\hat{R}^{-1} = I$ and $\hat{R}^{-1}\hat{R} = I$ (with $\hat{R}^{-1}$ given by (30)) are satisfied if the following (again equivalent) conditions hold

$$UU^{+} - VV^{+} = I,\; UV^{T} = VU^{T}, \tag{32}$$

$$U^{+}U - V^{T}\bar{V} = I,\; U^{T}\bar{V} = V^{+}U. \tag{33}$$

Hence $\|U\| \geq 1$ and U has an inverse. Then the identities

$$U^{-1}V = V^{T}\left(U^{-1}\right)^{T}, \quad \bar{V}U^{-1} = \left(U^{-1}\right)^{T}V^{+} \tag{34}$$

follow. Therefore the operators $U^{-1}V$ and $\bar{V}U^{-1}$ are symmetric. Moreover we obtain from (32) – (34)

$$I - \left(U^{-1}V\right)\left(U^{-1}V\right)^{+} = \left(U^{+}U\right)^{-1}, \tag{35}$$

$$I - \left(\bar{V}U^{-1}\right)^{+}\left(\bar{V}U^{-1}\right) = \left(UU^{+}\right)^{-1}. \tag{36}$$

The operator norms of $U^{-1}V$ and $\bar{V}U^{-1}$ therefore satisfy

$$\left\|U^{-1}V\right\|^2 = \left\|\bar{V}U^{-1}\right\|^2 = 1 - \|U\|^{-2} < 1. \tag{37}$$

The group element $\hat{R}(U,V) \in \widehat{S}p(\mathcal{H})$ maps $\mathcal{H}_{diag} \subset \mathcal{H} \times \mathcal{H}^*$ into itself

$$\hat{R}(U,V)\begin{pmatrix} f \\ f^* \end{pmatrix} = \begin{pmatrix} U & V \\ \bar{V} & \bar{U} \end{pmatrix}\begin{pmatrix} f \\ f^* \end{pmatrix} = \begin{pmatrix} Uf + Vf^* \\ \bar{U}f^* + \bar{V}f \end{pmatrix} \in \mathcal{H}_{diag}. \tag{38}$$

The operator $\hat{R}(U,V)$ is therefore uniquely determined by the following $\mathbb{R}$-linear mapping $R(U,V)$ on $\mathcal{H}$ (more precisely on $\mathcal{H}_{\mathbb{R}}$)

$$R(U,V)f = Uf + Vf^*. \tag{39}$$

The calculations presented above imply

Lemma 4 *The skew symmetric form (9) is invariant against the $\mathbb{R}$- linear mapping R on $\mathcal{H}_R$ if and only if R has the form (39) where U and V are bounded operators on $\mathcal{H}$, which satisfy the relations (32) and (33).*

In the sequel we often refer to (39) as the symplectic transformation. The product and the inverse follow from (29) and (30) as

$$R(U_2,V_2)R(U_1,V_1) = R(U_2U_1 + V_2\bar{V}_1, U_2V_1 + V_2\bar{U}_1), \tag{40}$$
$$R^{-1}(U,V) = R(U^+, -V^T). \tag{41}$$

The set of these transformations forms the group of symplectic transformations, which will be denoted by $Sp(\mathcal{H})$. The identity of the group is $R(I,0)$. In order to derive a unitary representation of this group on the Fock space $\mathcal{S}(\mathcal{H})$ an additional constraint is necessary if $\dim \mathcal{H}$ is infinite: The operator V has to be a Hilbert-Schmidt operator [7, 26]. This constraint is stable under the group operations (40) and (41).

Definition 3 *The group $Sp_2(\mathcal{H})$ is the subgroup of all transformations (39) $R(U,V) \in Sp(\mathcal{H})$ with a bounded operator $U \in \mathcal{L}(\mathcal{H})$ and a Hilbert-Schmidt operator $V \in \mathcal{L}_2(\mathcal{H})$.*

In [22] and [26] the elements of $Sp_2(\mathcal{H})$ are called restricted symplectic transformations, in [6] proper canonical transformations.

Let $U \in \mathcal{L}(\mathcal{H})$ be a unitary operator, then $R(U,0)$ is an isometric transformation in $Sp_2(\mathcal{H})$. Since $R(U,0)f = Uf$ for all $f \in \mathcal{H}_{\mathbb{R}}$ we simply write U for this transformation. In [26] Lemma 2.3 it has been derived that any symplectic transformation $R \in Sp_2(\mathcal{H})$ can be factorized in the form $R = U_1DU_2$. Thereby $U_{1,2}$ are two unitary transformations, and D is a real positive transformation in $Sp_2(\mathcal{H})$. This positive transformation has the form $D = R(\cosh A, \sinh A)$ with a real self-adjoint Hilbert-Schmidt operator A on $\mathcal{H}$.

4.2 Transformations of the Siegel disc

There is a non-linear representation of the restricted symplectic group by transformations on the Siegel disc, investigated by Siegel for the finite dimensional case [27]. Here we extend some of these results to the case of infinite dimensions.

From the definition of the Siegel disc follows that a Hilbert-Schmidt operator $Z = Z^T$ is an element of $\mathbf{D}_1$ if and only if

$$I - ZZ^+ > 0 \quad \text{(all eigenvalues strictly positive)}. \tag{42}$$

Lemma 5 *For all $R \in Sp_2(\mathcal{H})$ the transformation*

$$Z \to \widetilde{Z} = \zeta(R; Z) := (UZ + V)\left(\bar{U} + \bar{V}Z\right)^{-1} = \left(U^+ + ZV^+\right)^{-1}\left(V^T + ZU^T\right) \tag{43}$$

is an automorphism of the set $\mathbf{D}_1$. Thereby the group $Sp_2(\mathcal{H})$ acts transitively on $\mathbf{D}_1$.

Proof. For $R \in Sp_2(\mathcal{H})$ we have $V \in \mathcal{L}_2(\mathcal{H})$, and $\widetilde{Z}$ is a Hilbert-Schmidt operator. From (37) $\left\|U^{-1}V\right\|^2 = \left\|\bar{V}U^{-1}\right\|^2 = 1 - \|U\|^{-2} < 1$ and $|Z| < 1$ we know that $\|U^{-1}VZ\| < 1$, therefore the operator $U + VZ = U\left(I + U^{-1}VZ\right)$ is invertible. Hence

$$\begin{aligned}
&I - \widetilde{Z}\widetilde{Z}^+ = I - \left(U^+ + ZV^+\right)^{-1}\left(ZU^T + V^T\right)\left(\bar{U}Z^+ + \bar{V}\right)\left(U + VZ^+\right)^{-1} \\
&= \left(U^+ + ZV^+\right)^{-1}\left\{\left(U^+ + ZV^+\right)\left(U + VZ^+\right) - \left(ZU^T + V^T\right)\left(\bar{U}Z^+ + \bar{V}\right)\right\}\left(U + VZ^+\right)^{-1} \\
&= \left(U^+ + ZV^+\right)^{-1}\left\{I - ZZ^+\right\}\left(U + VZ^+\right)^{-1} > 0,
\end{aligned}$$

since $I - ZZ^+ > 0$.
The proof of the transitivity follows as in the finite dimensional case, see [27]. Let $Z \in \mathbf{D}_1$ then $I - ZZ^+ > 0$ and we can determine a $U \in \mathcal{L}(\mathcal{H})$ such that $U\left(I - ZZ^+\right)U^+ = I$. A special choice is $U = \left(I - ZZ^+\right)^{-\frac{1}{2}} \geq I$. The pair U and $V = UZ \in \mathcal{L}_2(\mathcal{H})$ satisfies the identities (32), and we easily derive $\zeta(R; 0) = U^{+-1}V^T = Z$. $\square$

The mapping ζ satisfies the rules

$$\begin{aligned}
\zeta(id; Z) &= Z \\
\zeta(R_2; \zeta(R_1; Z)) &= \zeta(R_2R_1; Z).
\end{aligned} \tag{44}$$

Hence $R \to \zeta(R; .)$ is a (nonlinear) representation of the group $Sp(\mathcal{H})$. If $R = R(U, 0)$ with a unitary operator U then ζ has the simple form $\zeta(R; Z) = UZU^T$.

5 Unitary representations of the symplectic group

A unitary ray representation of $Sp_2(\mathcal{H})$ in $\mathcal{S}(\mathcal{H})$ has the following properties:

$$\begin{gathered}
R \in Sp_2(\mathcal{H}) \longmapsto T(R) \text{ unitary operator on } \mathcal{S}(\mathcal{H}) \\
T(id) = I,\ T^{-1}(R) = T^+(R) = T\left(R^{-1}\right) \\
T(R_2)T(R_1) = \chi(R_2, R_1)T(R_2R_1) \\
\text{with } \chi(R_2, R_1) \in \mathbb{C},\ |\chi(R_2, R_1)| = 1.
\end{gathered} \tag{45}$$

In this section we construct a unitary ray representation by giving an explicit formula for $T(R)$ acting on ultracoherent states. As a first step $T(R)$ is defined as an isometric operator on the set of exponential vectors in Sect 5.1.1. This operator can be extended by linearity and continuity to a unitary operator on the Fock space. In Sect. 5.1.2 we derive an explicit formula for the action of $T(R)$ on ultracoherent vectors. In Sect. 5.1.3 we prove that $R \in Sp_2(\mathcal{H}) \longmapsto T(R)$ is a ray representation on the linear span of all ultracoherent vectors. Hence $T(R)$ is a unitary ray representation of the restricted symplectic group on $\mathcal{S}(\mathcal{H})$. Finally we prove in Sect. 5.2 that the operators $T(R)$ are Bogoliubov transformations, i.e. they generate homogeneous linear canonical transformations.

5.1 Representation of the group $Sp_2(\mathcal{H})$

5.1.1 Ansatz for coherent states

Let $R = R(U, V)$ be a symplectic transformation of the group $Sp_2(\mathcal{H})$ – i.e., $U \in \mathcal{L}(\mathcal{H})$ and $V \in \mathcal{L}_2(\mathcal{H})$ – then $|U| := \sqrt{UU^+} = \sqrt{I + VV^+} \geq I$ has the property $|U| - I \in \mathcal{L}_1(\mathcal{H})$ and the determinants $\det|U| \geq 1$ and $\det|U|^{-1} = (\det|U|)^{-1}$ are well defined.

The representation $T(R)$ of the group $Sp_2(\mathcal{H})$ is now defined on the set of exponential vectors by

$$T(R)\exp f := (\det|U|)^{-\frac{1}{2}}\,\Phi\left(U^{+-1}V^T,\, U^{+-1}f\right)\exp\left(-\frac{1}{2}\left\langle f \mid V^+U^{+-1}f\right\rangle\right). \tag{46}$$

Since V is a Hilbert-Schmidt operator, the relations (34) and (37) imply that the mapping $U^{+-1}V^T = V\bar{U}^{-1}$ is an element of the Siegel unit disc. Hence the ultracoherent vector is an element of the Fock space $\mathcal{S}(\mathcal{H})$. In the special case of a unitary transformation $R(U, 0)$ with U unitary the ansatz (46) has the simple form $T(R)\exp f = \exp(Uf)$ such that, see (5),

$$T(R(U,0)) = \Gamma(U). \tag{47}$$

The transformation $T(R)$ defined in (46) can then be extended by linearity onto the linear span $\mathcal{S}_{coh}(\mathcal{H})$ of all exponential vectors. Thereby the identity of the group $R(I_{\mathcal{H}}, 0)$ is mapped onto the unit operator on $\mathcal{S}_{coh}(\mathcal{H})$. In the subsequent part of this Section it is shown that $R \in Sp_2(\mathcal{H}) \to T(R)$ is actually a unitary ray representation of the group $Sp_2(\mathcal{H})$ on the Fock space $\mathcal{S}(\mathcal{H})$.

Lemma 6 *The operator (46) has a unique extension to a linear unitary mapping on* $\mathcal{S}(\mathcal{H})$.

Proof. For the proof of this statement we calculate the inner product

$$\begin{aligned}(T(R)\exp f \mid T(R)\exp g) = \det|U|^{-1}\exp\left(-\tfrac{1}{2}\overline{\langle f \mid V^+U^{+-1}f\rangle} - \tfrac{1}{2}\left\langle g \mid V^+U^{+-1}g\right\rangle\right)\\ \times\left(\Phi\left(U^{+-1}V^T,\, U^{+-1}f\right) \mid \Phi\left(U^{+-1}V^T,\, U^{+-1}g\right)\right).\end{aligned}$$

The inner product

$$\begin{aligned}\left(\Phi\left(U^{+-1}V^T,\, U^{+-1}f\right) \mid \Phi\left(U^{+-1}V^T,\, U^{+-1}g\right)\right) = \det\left(I - U^{+-1}V^T\bar{V}U^{-1}\right)^{-\frac{1}{2}}\\ \times\exp\left(\tfrac{1}{2}\left\langle U^{+-1}g \mid D\,U^{+-1}g\right\rangle + \tfrac{1}{2}\overline{\langle U^{+-1}f \mid D\,U^{+-1}f\rangle}\right)\\ \times\exp\left\langle U^{T-1}f^* \mid (I - U^{+-1}V^T\bar{V}U^{-1})^{-1}\,U^{+-1}g\right\rangle,\end{aligned}$$

follows from (14). Thereby D is given by $D = \bar{V}U^{-1}(I - U^{+-1}V^T\bar{V}U^{-1})^{-1}$. Since $I - U^{+-1}V^T\bar{V}U^{-1} \overset{(36)}{=} (UU^+)^{-1}$ we have $D = \bar{V}U^+$ and $\det\left(I - U^{+-1}V^T\bar{V}U^{-1}\right) = (\det|U|)^{-2}$. We finally obtain

$$(T(R)\exp f \mid T(R)\exp g) = (\exp f \mid \exp g) \tag{48}$$

for all $f, g \in \mathcal{H}$. Lemma 2 then implies that $T(R)$ can be extended to an isometric mapping, which we denote by the same symbol.

The calculation of $(\exp g \mid T^+(R)\exp f) = (T(R)\exp g \mid \exp f)$ using (14) yields

$$T^+(R)\exp f = (\det|U|)^{-\frac{1}{2}}\,\Phi\left(-U^{-1}V, U^{-1}f\right)\exp\frac{1}{2}\left\langle f \mid \bar{V}U^{-1}f\right\rangle. \tag{49}$$

Inserting (30) into (46) we obtain

$$T(R^{-1}) = T^+(R), \tag{50}$$

first on $\mathcal{S}_{coh}(\mathcal{H})$ and by continuity on $\mathcal{S}(\mathcal{H})$. Since $T(R^{-1})$ is isometric the operator $T^+(R)$ is also an isometric mapping. Hence $T(R)$ is unitary. □

Remark 2 *If $\mathcal{H}$ is finite dimensional, we can use the determinant* $\det U$ *(instead of* $\det|U|$*) in the definition (46).*

Remark 3 *The representations of the finite dimensional symplectic group have been investigated in [5, 13, 16] using the complex wave representation of the Fock space. These authors calculate the kernel function* $(\exp g \mid T(R)\exp f)$ *of the operator* $T(R)$*. The ansatz (46) is motivated by these papers.*

5.1.2 Extension to ultracoherent vectors

With help of the relation (14) we can derive a closed formula for $T(R)$ operating on ultracoherent vectors

$$\begin{aligned}
&(\exp z \mid T(R)\Phi(Z,\, f)) = (T^+(R)\exp z \mid \Phi(Z,\, f)) \\
&\overset{(49)}{=} (\det|U|)^{-\frac{1}{2}}\left(\Phi\left(-U^{-1}V,\, U^{-1}z\right) \mid \Phi(Z,\, f)\right)\exp\tfrac{1}{2}\overline{\left\langle z \mid \bar{V}U^{-1}z\right\rangle} \\
&\overset{(14)}{=} (\det|U|)^{-\frac{1}{2}}\left(\det(I + ZV^+U^{+-1})\right)^{-\frac{1}{2}}\exp\left(\tfrac{1}{2}\left\langle z^* \mid V\bar{U}^{-1}z^*\right\rangle + \tfrac{1}{2}\left\langle \bar{U}^{-1}z^* \mid C\bar{U}^{-1}z^*\right\rangle\right) \\
&\quad\times\exp\left(\left\langle \bar{U}^{-1}z^* \mid (I + ZV^+U^{+-1})^{-1}f\right\rangle + \tfrac{1}{2}\left\langle f \mid Df\right\rangle\right)
\end{aligned} \tag{51}$$

with the operators

$$\begin{aligned}
C &= (I + ZV^+U^{+-1})^{-1}Z, \\
D &= -V^+(U^+ + ZV^+)^{-1}.
\end{aligned} \tag{52}$$

Since $U^{+-1}C\bar{U}^{-1} + V\bar{U}^{-1} = (U^+ + ZV^+)^{-1}(V^T + ZU^T) = \zeta(R;Z)$, we finally obtain

$$\begin{aligned}
&T(R)\,\Phi(Z,\, f) = (\det|U|)^{-\frac{1}{2}}\left(\det(I + ZV^+U^{+-1})\right)^{-\frac{1}{2}} \\
&\times\Phi\left(\zeta(R;Z),\, (U^+ + ZV^+)^{-1}f\right)\exp\left(-\tfrac{1}{2}\left\langle f \mid V^+(U^+ + ZV^+)^{-1}f\right\rangle\right).
\end{aligned} \tag{53}$$

5.1.3 The multiplication law

In the next step we prove

$$T(R_2)T(R_1) = \chi(R_2, R_1)T(R_3) \quad \text{if } R_2R_1 = R_3 \tag{54}$$

with a multiplier $\chi(R_2, R_1) \in \mathbb{C}$, $|\chi(R_2, R_1)| = 1$.

Let

$$\begin{aligned} Z_1 &= \zeta(R_1; Z) = (U_1^+ + ZV_1^+)^{-1}(V_1^T + ZU_1^T) \\ Z_2 &= \zeta(R_2; Z_1) = \zeta(R_3; Z) \end{aligned} \tag{55}$$

see (44), then we obtain from (53)

$$\begin{aligned} &T(R_1)\Phi(Z, f) = (\det|U_1|)^{-\frac{1}{2}} \left(\det(I + ZV_1^+U_1^{+-1})\right)^{-\frac{1}{2}} \\ &\times\Phi\left(Z_1, (U_1^+ + ZV_1^+)^{-1}f\right) \exp\left(-\tfrac{1}{2}\left\langle f \mid V_1^+(U_1^+ + ZV_1^+)^{-1}f\right\rangle\right) \end{aligned}$$

and

$$\begin{aligned} &T(R_2)T(R_1)\Phi(Z, f) \\ &= (\det|U_2|)^{-\frac{1}{2}} \left(\det(I + Z_1V_2^+U_2^{+-1})\right)^{-\frac{1}{2}} (\det|U_1|)^{-\frac{1}{2}} \left(\det(I + ZV_1^+U_1^{+-1})\right)^{-\frac{1}{2}} \\ &\times\Phi\left(Z_2, (U_2^+ + Z_1V_2^+)^{-1}(U_1^+ + ZV_1^+)^{-1}f\right) \\ &\times\exp\left(-\tfrac{1}{2}\left\langle(U_1^+ + ZV_1^+)^{-1}f \mid V_2^+(U_2^+ + Z_1V_2^+)^{-1}(U_1^+ + ZV_1^+)^{-1}f\right\rangle\right) \\ &\times\exp\left(-\tfrac{1}{2}\left\langle f \mid V_1^+(U_1^+ + ZV_1^+)^{-1}f\right\rangle\right). \end{aligned} \tag{56}$$

The tensor of second degree in the exponent immediately follows as $Z_2 = \zeta(R_2; Z_1) = \zeta(R_3; Z)$. The operator product $(U_2^+ + Z_1V_2^+)^{-1}(U_1^+ + ZV_1^+)^{-1}$ is calculated with

$$\begin{aligned} &U_2^+ + Z_1V_2^+ = (U_1^+ + ZV_1^+)^{-1}\left((U_1^+ + ZV_1^+)U_2^+ + (V_1^T + ZU_1^T)V_2^+\right) \\ &= (U_1^+ + ZV_1^+)^{-1}\left(U_1^+U_2^+ + ZV_1^+U_2^+ + V_1^TV_2^+ + ZU_1^TV_2^+\right) = (U_1^+ + ZV_1^+)^{-1}(U_3^+ + ZV_3^+) \end{aligned}$$

as

$$(U_2^+ + Z_1V_2^+)^{-1}(U_1^+ + ZV_1^+)^{-1} = (U_3^+ + ZV_3^+)^{-1}. \tag{57}$$

Hence we obtain

$$\begin{aligned} T(R_2)T(R_1)\Phi(Z, f) &= \chi\, T(R_3)\Phi(Z, f) \\ &\times\exp\left(-\frac{1}{2}\left(\alpha_{12} - \left\langle f \mid V_3^+(U_3^+ + ZV_3^+)^{-1}f\right\rangle\right)\right) \end{aligned} \tag{58}$$

with

$$\chi = \sqrt{\frac{\det|U_3|}{\det|U_1|\det|U_2|}}\sqrt{\frac{\det(I + ZV_3^+U_3^{+-1})}{\det(I + ZV_1^+U_1^{+-1})\det(I + Z_1V_2^+U_2^{+-1})}} \tag{59}$$

and

$$\begin{aligned} \alpha_{12} = &\left\langle(U_1^+ + ZV_1^+)^{-1}f \mid V_2^+(U_2^+ + Z_1V_2^+)^{-1}(U_1^+ + ZV_1^+)^{-1}f\right\rangle \\ &+\left\langle f \mid V_1^+(U_1^+ + ZV_1^+)^{-1}f\right\rangle. \end{aligned} \tag{60}$$

Now we choose $Z = 0$. Then (59) simplifies to

$$\chi(R_2, R_1) = \sqrt{\frac{\det|U_3|}{\det|U_1|\det|U_2|\det\left(U_1^{+-1}U_3^+U_2^{+-1}\right)}}, \tag{61}$$

which depends only on the group elements. In the next step we evaluate (60) for $Z = 0$

$$\begin{aligned}\alpha_{12} &= \left\langle U_1^{+-1} f \mid V_2^+ (U_2^+ + U_1^{+-1} V_1^T V_2^+)^{-1} U_1^{+-1} f \right\rangle + \left\langle f \mid V_1^+ U_1^{+-1} f \right\rangle \\ &= \left\langle f \mid \left(\bar{U}_1^{-1} V_2^+ (U_2^+ + U_1^{+-1} V_1^T V_2^+)^{-1} U_1^{+-1} + V_1^+ U_1^{+-1} \right) f \right\rangle .\end{aligned}$$

Since $(U_2^+ + U_1^{+-1} V_1^T V_2)^{-1} = \left(U_1^+ U_2^+ + V_1^T V_2 \right)^{-1} U_1^+$ we have

$$\begin{aligned}&\bar{U}_1^{-1} V_2^+ (U_2^+ + U_1^{+-1} V_1^T V_2^+)^{-1} U_1^{+-1} + V_1^+ U_1^{+-1} \\ &= \left(V_1^+ U_1^{+-1} \left(U_1^+ U_2^+ + V_1^T V_2^+ \right) + \bar{U}_1^{-1} V_2^+ \right) \left(U_1^+ U_2^+ + V_1^T V_2^+ \right)^{-1} \\ &\overset{(29),(34)}{=} \left(V_1^+ U_2^+ + \bar{U}_1^{-1} (\bar{V}_1 V_1^T + I) V_2^+ \right) U_3^{+-1} \\ &\overset{(32)}{=} \left(V_1^+ U_2^+ + \bar{U}_1^{-1} \bar{U}_1 U_1^T V_2^+ \right) U_3^{+-1} = \left(V_1^+ U_2^+ + U_1^T V_2^+ \right) U_3^{+-1} \overset{(29)}{=} V_3^+ U_3^{+-1},\end{aligned}$$

such that

$$\alpha_{12} = \left\langle f \mid V_3^+ U_3^{+-1} f \right\rangle . \tag{62}$$

The identity (58) together with (61) and (62) imply

$$T(R_2) T(R_1) \exp f = \chi(R_1, R_2)\, T(R_3) \exp f \tag{63}$$

for all $f \in \mathcal{H}$. But then (54) is true as an operator identity. Since we already know that the operators $T(R)$ are unitary, the modulus of χ is $|\chi(R_1, R_2)| = 1$.

Remark 4 *The identity (50) implies* $\chi(R, R^{-1}) = \chi(R^{-1}, R) = 1$.

Remark 5 *Since* $\det |K U K^{-1}| = \det |U|$ *the definition (46) implies that*

$$T(K R K^{-1}) = \Gamma(K) T(R) \Gamma(K^+) \tag{64}$$

holds for all unitary transformations K *and all* $R \in Sp_2(\mathcal{H})$ *without additional phase factor.*

Remark 6 *If* $\mathcal{H}$ *is finite dimensional, the determinant* $\det U$ *(instead of* $\det |U|$*) can be used in the definition (46). Then the multiplier* $\chi(R_1, R_2)$ *takes only the values* ± 1*, see [5].*

5.2 Weyl operators and Bogoliubov transformations

In this Section we prove that the operators $T(R)$ are Bogoliubov transformations as defined in Sect.3.2.

Given $f, g \in \mathcal{H}$ and $R(U, V) \in Sp_2(\mathcal{H})$ we first calculate

$$\begin{aligned}&T(R) W(f) \exp g \overset{(18)}{=} \exp\left(- \langle f^* \mid g \rangle - \tfrac{1}{2} \|f\|^2 \right) T(R) \exp(f + g) \\ &= (\det |U|)^{-\frac{1}{2}}\, \Phi\left(U^{+-1} V^T,\, U^{+-1}(f + g) \right) \\ &\times \exp\left(- \langle f^* \mid g \rangle - \tfrac{1}{2} \|f\|^2 - \tfrac{1}{2} \langle f + g \mid V^+ U^{+-1} (f + g) \rangle \right) .\end{aligned} \tag{65}$$

On the other hand $W(Rf)T(R)\exp g$ follows as

$$\begin{aligned}
& W(Rf)T(R)\exp g \overset{(46)}{=} \det|U|^{-\frac{1}{2}} \exp\left(-\tfrac{1}{2}\left\langle g \mid V\bar{U}^{-1}g\right\rangle\right) \\
& \quad \times W(Uf+Vf^*)\Phi\left(V\bar{U}^{-1},\, U^{+-1}g\right) \\
& \overset{(20)}{=} (\det|U|)^{-\frac{1}{2}} \exp\left(-\tfrac{1}{2}\left\langle g \mid V^+U^{+-1}g\right\rangle - \left\langle \bar{U}f^* + \bar{V}f \mid U^{+-1}g\right\rangle\right) \\
& \quad \times \exp\left(+\tfrac{1}{2}\left\langle \bar{U}f^* + \bar{V}f \mid V\bar{U}^{-1}(\bar{U}f^* + \bar{V}f) - (Uf+Vf^*)\right\rangle\right) \\
& \quad \times \Phi\left(U^{+-1}V^T,\, Uf+Vf^* + U^{+-1}g - V\bar{U}^{-1}(\bar{U}f^*+\bar{V}f)\right) \\
& = (\det|U|)^{-\frac{1}{2}}\, \Phi\left(U^{+-1}V^T,\, U^{+-1}(f+g)\right) \\
& \times \exp(-\left\langle f^* \mid g\right\rangle - \tfrac{1}{2}\|f\|^2 - \tfrac{1}{2}\left\langle f+g \mid V^+U^{+-1}(f+g)\right\rangle).
\end{aligned} \tag{66}$$

The last identity follows using (34) and (35). Hence we have derived the identity $T(R)W(f) = W(Rf)T(R)$ on $\mathcal{S}_{coh}(\mathcal{H})$. But then (24) is true on $\mathcal{S}(\mathcal{H})$ for all homogeneous canonical transformations $S = T(R)$, $R \in Sp_2(\mathcal{H})$.

6 Concluding remarks

In this paper we have given a self-contained construction of the unitary representation of the (in)finite dimensional symplectic group on the Fock space. The operators are first defined on the linear span of all exponential vectors (coherent states) and then extended onto the minimal linear set, which is closed under the action of Weyl operators and of homogeneous linear canonical transformations. Actually, the calculations of the paper show that any ultracoherent vector $\Phi(A,f)$ has a representation $\Phi(A,f) = const \cdot W(h)T(R)\,1_{vac}$ where 1_{vac} is the vacuum state of the Fock space, $W(h)$ is a Weyl operator and $T(R)$ is a Bogoliubov transformation. This follows from the transitivity of the action of $Sp_2(\mathcal{H})$ on the Siegel disc, see Lemma 5, and from the formulas (20) and (75). The states $T(R)\,1_{vac}$ are the squeezed vacua of quantum optics. Our presentation of canonical transformations is therefore useful for applications in quantum optics and in quantum computation [3, 14, 15].

The constructions given in Sect. 5.1 are independent from the explicit representation of the Fock space. Nevertheless we would like to mention two of these representations:

1. The complex wave representation or Bargmann-Fock representation uses a reproducing kernel Hilbert space, see Remark 1. The operators $T(R)$ have integral kernels $(\exp g \mid T(R)\exp f)$ also in the infinite dimensional case. These kernels can easily be calculated from (46). They differ from the kernels for the finite dimensional group as given in [5, 13, 16] only by the choice of the determinant, see Remark 2.

2. The real wave representation or Wiener-Segal representation, see [2, 21], is closely related to the self-adjoint canonical field and momentum variables. To transfer the method of Sect 5.1 to this representation one has to use the real form of the symplectic group, and the Siegel disc has to be substituted by the Siegel upper half plane (operators of $\mathcal{L}_{2sym}$ with a positive imaginary part). The ultracoherent vectors are Gaussian functions in the position variable. Also in this case it is possible to implement Bogoliubov transformations by integral transforms [18, 19].

The explicit construction of $T(R)$ on a dense domain of definition – in this case on ultracoherent vectors – has some advantage for the investigation of one-parameter subgroups and their generators. In the infinite dimensional case the group $Sp_2(\mathcal{H})$ is still a topological group [26], but the one-parameter subgroups may have unbounded generators. A Lie group structure has been imposed only under the additional restriction to bounded generators, see Chapter 3 of [22]. For applications in quantum field theory one would like to consider subgroups with unbounded generators. Take as example a free field with an unbounded positive one-particle Hamiltonian $M = \bar{M}$. The domain of definition of M is the dense set $\mathcal{D}(M) \subset \mathcal{H}$. With $U_0(t) = \exp(-iMt)$ we denote the corresponding unitary group on $\mathcal{H}$. The linear set $\mathcal{D}_F = span\,\{\exp f \mid f \in \mathcal{D}(M)\}$ is dense in $\mathcal{S}(\mathcal{H})$, and the free field Hamiltonian $H_F = d\Gamma(M)$ is defined on $\mathcal{D}_F$ by $d\Gamma(M)\exp f = Mf \vee \exp f$. The unitary group $\exp(-iH_Ft) = \Gamma(U_0(t))$ is a one parameter subgroup of proper canonical transformations. If we now submit the free field to a homogeneous canonical transformation $S = T(R_1)$ with $R_1 = R(U_1, V_1) \in Sp_2(\mathcal{H})$, then the resulting unitary group in $\mathcal{S}(\mathcal{H})$ is $S\exp(-iH_Ft)S^{-1} = T(R_2(t))$ with $R_2(t) = R(U_1U_0(t)U_1^+ - V_1U_0(-t)V_1^+, -U_1U_0(t)V_1^T + V_1U_0(-t)U_1^T)$. The generator of this group is the Hamiltonian of the transformed free field. Formal differentiation of $T(R_2(t))$ to obtain the Hamiltonian is misleading, as expressions like $U_1MU_1^+ + V_1MV_1^+$ might not be defined because M is unbounded. But our equation (46) gives an explicit formula for a domain of analytic vectors for this Hamiltonian within the set of ultracoherent vectors: $\mathcal{D} = T(R_1)\mathcal{D}_F$.

A related application is the investigation of the representation of general one-parameter subgroups of the symplectic group. The complexity of this problem can be seen from the paper [12], where results on the basis of the white noise calculus have been derived.

A Calculations for ultracoherent vectors

A.1 Norm estimates for the tensor product

The symmetric tensor product is well defined on the algebra $\mathcal{S}_{fin}(\mathcal{H})$. To extend it to a larger space we introduce a family of Hilbert norms [20]. For Let $F = \sum_n F_n \in \mathcal{S}_{fin}(\mathcal{H})$ with $F_n \in \widehat{\mathcal{H}^{\vee n}}$ then the norm $\|F\|_{(\alpha)}$, $\alpha > 0$, is defined by

$$\|F\|_{(\alpha)}^2 = \sum_n \alpha^{-2n} \|F_n\|_n^2 \,. \tag{67}$$

Thereby $\|\,.\,\|_n$ is the Hilbert norm of $\widehat{\mathcal{H}^{\vee n}}$ introduced in Sect. 2.1. The completion of $\mathcal{S}_{fin}(\mathcal{H})$ with respect to the norm (67) is called $\mathcal{S}^{(\alpha)}(\mathcal{H})$. We obviously have $\mathcal{S}^{(1)}(\mathcal{H}) = \mathcal{S}(\mathcal{H})$ and

$$\begin{aligned} &\|F\| = \|F\|_{(1)} \le \|F\|_{(\alpha)} \text{ if } \alpha \in (0,1]\,, \\ &\|F_n\|_n \le \alpha^n \|F\|_{(\alpha)} \text{ if } \alpha > 0. \end{aligned} \tag{68}$$

Lemma 7 *The following statements are true:*

1. *For $\alpha \in (0,1]$ the algebras $\mathcal{S}_{fin}(\mathcal{H})$ and $\mathcal{S}_{coh}(\mathcal{H})$ are dense linear subsets of $\mathcal{S}^{(\alpha)}(\mathcal{H})$, and $\mathcal{S}^{(\alpha)}(\mathcal{H})$ is a dense linear subset of $\mathcal{S}(\mathcal{H})$.*

2. *Let $A \in \mathbf{D}_1$ then $\exp \Omega(A) \in \mathcal{S}^{(\alpha)}(\mathcal{H})$ with $\alpha \in (\|A\|, 1]$.*

3. *Let $0 < \alpha, \beta, \gamma < 1$ with $\alpha + \beta < \gamma \leq 1$, then there exists a constant $c_{\alpha\beta\gamma}$ such that*

$$\|F \vee G\|_{(\gamma)} \leq c_{\alpha\beta\gamma} \|F\|_{(\alpha)} \|G\|_{(\beta)} \tag{69}$$

for all F, $G \in \mathcal{S}_{fin}(\mathcal{H})$.

Proof. 1) The norm (67) of an exponential vector $\exp f$, $f \in \mathcal{H}$, is $\|\exp f\|_{(\alpha)}^2 = \|\exp(\alpha^{-1} f)\|^2$. Hence $\|\exp f\|_{(\alpha)} < \infty$ for all $\alpha > 0$ and $\mathcal{S}_{coh}(\mathcal{H})$ is a subset of $\mathcal{S}^{(\alpha)}(\mathcal{H})$, $\alpha \in (0, 1]$. Since $\mathcal{S}_{coh}(\mathcal{H})$ is dense in $\mathcal{S}(\mathcal{H})$, the identity (67) implies that $\mathcal{S}_{coh}(\mathcal{H})$ is dense in $\mathcal{S}^{(\alpha)}(\mathcal{H})$ in the topology (67). The other statements of 1) are obvious.

2) Assume $A \in \mathbf{D}_1$ then for $0 < \lambda < \|A\|^{-1}$ also λA is an element of $\mathbf{D}_1$. Hence the series $\exp \Omega(\lambda A) = \sum_n \frac{\lambda^n}{n!} (\Omega(A))^{\vee n}$ converges within $\mathcal{S}(\mathcal{H})$ and $\left\| \frac{\lambda^n}{n!} (\Omega(A))^{\vee n} \right\|_{2n}^2 \leq \|\exp \Omega(\lambda A)\|^2 = C < \infty$ follows. The estimate $\left\| \frac{1}{n!} (\Omega(A))^{\vee n} \right\|_{2n}^2 \leq C \lambda^{-2n}$ implies $\|\exp \Omega(A)\|_{(\alpha)}^2 = \sum_n \alpha^{-2n} \left\| \frac{1}{n!} (\Omega(A))^{\vee n} \right\|_{2n}^2 \leq C \sum_n (\alpha\lambda)^{-2n} < \infty$ for all $\alpha > \lambda^{-1} > \|A\|$.

3) Let $F = \sum_n F_n$ and $G = \sum_n G_n$ be two elements of $\mathcal{S}_{fin}(\mathcal{H})$ with $F_n, G_n \in \widehat{\mathcal{H}^{\vee n}}$, then (68) implies $\|F_n\|_n \leq \alpha^n \|F\|_{(\alpha)}$ and $\|G_n\|_n \leq \beta^n \|G\|_{(\beta)}$ with arbitrary $\alpha, \beta > 0$. The symmetric tensor product $F \vee G$ is then calculated as $F \vee G = \sum_n H_n$ with $H_n = \sum_{m=0}^n F_m \vee G_{n-m} \in \widehat{\mathcal{H}^{\vee n}}$. From (3) and (68) we have

$$\begin{aligned} \left\|\sum_{m=0}^n F_m \vee G_{n-m}\right\|_n &\leq \sum_{m=0}^n \sqrt{\begin{pmatrix} n \\ m \end{pmatrix}} \|F_m\|_m \|G_{n-m}\|_{n-m} \\ &\leq \|F\|_{(\alpha)} \|G\|_{(\beta)} \sum_{m=0}^n \begin{pmatrix} n \\ m \end{pmatrix} \alpha^m \beta^{n-m} = \|F\|_{(\alpha)} \|G\|_{(\beta)} (\alpha + \beta)^n. \end{aligned}$$

Hence $\|F \vee G\|_{(\gamma)}^2 \leq \|F\|_{(\alpha)}^2 \|G\|_{(\beta)}^2 \sum_n \gamma^{-2n} (\alpha + \beta)^{2n}$, and the upper bound (69) follows for all $\alpha, \beta, \gamma > 0$ with $\alpha + \beta < \gamma \leq 1$. □

The third statement of this Lemma implies

Corollary 1 *Let α, β, γ be strictly positive numbers with $\alpha + \beta < \gamma \leq 1$, then the symmetric tensor product $F, G \to F \vee G = G \vee F$ is a $\mathbb{C}$-bilinear continuous mapping from $\mathcal{S}^{(\alpha)}(\mathcal{H}) \times \mathcal{S}^{(\beta)}(\mathcal{H})$ into $\mathcal{S}^{(\gamma)}(\mathcal{H})$.*

If $A \in \mathbf{D}_1$ then we know from the second statement of Lemma 7 that $\exp \Omega(A) \in \mathcal{S}^{(\alpha)}(\mathcal{H})$ for some $\alpha \in (\|A\|, 1)$. On the other hand the exponential vector $\exp f$, $f \in \mathcal{H}$, is an element of $\mathcal{S}^{(\beta)}(\mathcal{H})$ with $0 < \beta < 1 - \alpha$. The Corollary then implies that the symmetric tensor product $\exp \Omega(A) \vee \exp f = \exp f \vee \exp \Omega(A)$ is an element of the Fock space, which depends continuously on its factors. The ultracoherent vectors (13) are therefore defined as elements of $\mathcal{S}(\mathcal{H})$ for $A \in \mathbf{D}_1$ and $f \in \mathcal{H}$. Moreover, since $f \in \mathcal{H} \to \exp f$ and $A \in \mathbf{D}_1 \to \exp \Omega(A)$ are holomorphic functions, the ultracoherent vector (13) $\Phi(A, f) = \exp \Omega(A) \vee \exp f$ is an analytic function of $A \in \mathbf{D}_1$ and $f \in \mathcal{H}$.

A.2 Identities

Let F and G be two elements of $\mathcal{S}_{coh}(\mathcal{H})$, then the identity

$$(\exp z \mid F \vee G) = (\exp z \mid F)\,(\exp z \mid G) \tag{70}$$

follows. For the proof it is sufficient to choose the exponential vectors $F = \exp f$ and $G = \exp g$. This identity can be extended by continuity to $F \in \mathcal{S}^{(\alpha)}(\mathcal{H})$ and $G \in \mathcal{S}^{(\beta)}(\mathcal{H})$ if $\alpha, \beta > 0$ and $\alpha + \beta < 1$.

The inner product $(\exp z \mid \exp \Omega(A))$ is calculated by evaluation of the power series using the product rule (70) as

$$(\exp z \mid \exp \Omega(A)) = \exp \frac{1}{2} (z \vee z \mid \Omega(A)) = \exp \frac{1}{2} \langle z^* \mid Az^* \rangle . \tag{71}$$

Further application of (70) yields

$$(\exp z \mid \exp \Phi(A, f)) = (\exp z \mid \exp \Omega(A) \vee \exp f) = \exp \left(\frac{1}{2} \langle z^* \mid Az^* \rangle + \langle z^* \mid f \rangle \right) . \tag{72}$$

The inner product (72) can be used to determine the action of the Weyl operator $W(h)$ on the ultracoherent vector $\Phi(A, f)$. We have

$$\begin{aligned}
&(\exp z \mid W(h)\Phi(A, f)) = (W(-h) \exp z \mid \Phi(A, f)) \\
&\overset{(18)}{=} \mathrm{e}^{-\frac{1}{2}\|h\|^2 + \frac{1}{2}\langle h^* | Ah^* \rangle - \langle h^* | f \rangle} \mathrm{e}^{\langle f + h - Ah^* | z^* \rangle + \frac{1}{2}\langle z^* | Az^* \rangle} \\
&= \mathrm{e}^{-\frac{1}{2}\|h\|^2 + \frac{1}{2}\langle h^* | Ah^* \rangle - \langle h^* | f \rangle} (\exp z \mid \Phi(A, f + h - Ah^*))
\end{aligned}$$

which implies the relation (20).

The restriction of (20) to $\Phi(A, 0)$ is

$$W(h) \exp \Omega(A) = \mathrm{e}^{\frac{1}{2}\langle h^* | Ah^* - h \rangle} \Phi(A, h - Ah^*). \tag{73}$$

Since the norm of $\exp \Omega(A)$ is known and $W(h)$ is unitary, we use (73) to calculate the norm of the exponential vectors.

Any operator $A \in \mathbf{D}_1$ satisfies $\|A\| < 1$, therefore $(I - A^+A)^{-1}$ and $(I - AA^+)^{-1}$ are bounded operators. Let f be a vector in $\mathcal{H}$, then

$$h = (I - AA^+)^{-1} f + A(I - A^+A)^{-1} f^* \tag{74}$$

is an element of $\mathcal{H}$, which satisfies $h - Ah^* = f$. With this vector h we obtain from (73)

$$W(h) \exp \Omega(A) = \Phi(A, f) \exp \left(\frac{1}{2} \left\langle f \mid (I - A^+A)^{-1} f^* + A^+(I - AA^+)^{-1} f \right\rangle \right) . \tag{75}$$

Since the Weyl operator is unitary, we know from (12)

$$(W(h) \exp \Omega(A) \mid W(h) \exp \Omega(A)) = (\exp \Omega(A) \mid \exp \Omega(A)) = \det \left(I - A^+A \right)^{-\frac{1}{2}} . \tag{76}$$

Substituting (75) into this identity we obtain

$$
\begin{aligned}
&(\Phi(A,f) \mid \Phi(A,f)) = \det\left(I - A^+A\right)^{-\frac{1}{2}} \\
&\times \exp\left(-\tfrac{1}{2}\left\langle f^* \mid A(I-A^+A)^{-1}f^*\right\rangle + \left\langle f^* \mid (I-AA^+)^{-1}\, f\right\rangle + \tfrac{1}{2}\left\langle f \mid A^+(I-AA^+)^{-1}\, f\right\rangle\right).
\end{aligned} \tag{77}
$$

The inner product $\varphi(A,f;B,g) := (\Phi(A,f) \mid \Phi(B,g))$ of the exponential vectors $\Phi(A,f)$ and $\Phi(B,g)$ is analytic in B and g, and it is antianalytic in A and in f. The function $\varphi(A,f;B,g)$ is uniquely determined by its values on the diagonals $A=B$ and $f=g$. From (77) we then obtain

$$
\begin{aligned}
&\varphi(A,f;B,g) = \det\left(I - A^+B\right)^{-\frac{1}{2}} \\
&\times \exp\left(-\tfrac{1}{2}\left\langle f^* \mid C\, f^*\right\rangle + \left\langle f^* \mid (I-BA^+)^{-1}\, g\right\rangle + \tfrac{1}{2}\left\langle g \mid D\, g\right\rangle\right),
\end{aligned}
$$

where the operators C and D are defined in (15). Hence we have derived the inner product (14) of two ultracoherent vectors.

Remark 7 *With the diagonalization technique of [17] one can derive that* $\lim_{J\to\infty}\sum_{j=0}^{J}\frac{1}{j!}\left(\Omega(A)+f\right)^{\vee j}$ *is norm convergent for* $A \in \mathbf{D}_1$ *and* $f \in \mathcal{H}$. *We denote the limit by* $\exp\left(\Omega(A)+f\right)$. *Since* $\sum_{k=0}^{\infty}\frac{1}{k!}\left(\exp z \mid \left(\Omega(A)+f\right)^{\vee k}\right)$ $= \sum_{m,n=0}^{\infty}\frac{1}{m!n!}\left(\exp z \mid (\Omega(A))^{\vee m} \vee f^{\vee n}\right)$ *for all* $z \in \mathcal{H}$, *we obtain* $\exp\left(\Omega(A)+f\right) = \exp\Omega(A) \vee \exp f$. *For the proofs given in this paper the definition* $\Phi(A,f) = \exp\Omega(A) \vee \exp f$ *is sufficient. We have not used the identification with* $\exp\left(\Omega(A)+f\right)$.

A.3 Series expansion

In this Appendix we give a supplementary proof for the inner products (12) and (14) using a series expansion.

For $H \in \mathcal{S}_2(\mathcal{H})$ we obtain from (3) $\left\|H^{\vee 2n}\right\| \le \frac{(2n)!}{2^n}\left\|H\right\|^{2n}$. Hence the exponential series $\exp H = 1 + H + \frac{1}{2!}H^{\vee 2} + \frac{1}{3!}H^{\vee 3} + \ldots$ converges within $\mathcal{S}(\mathcal{H})$ with the norm estimate $\left\|\exp H\right\|^2 \le \sum_{n=0}^{\infty}\left(\frac{1}{n!}\right)^2\frac{(2n)!}{2^n}\left\|H\right\|^{2n} < \infty$ if $\left\|H\right\|^2 < 1/2$. The mapping $H \in \mathcal{S}_2(\mathcal{H}) \to \exp H \in \mathcal{S}(\mathcal{H})$ is therefore an analytic function within the open ball $\left\{H \mid \|H\| < 1/\sqrt{2}\right\}$. As a consequence of Lemma 3 we know that $\exp\Omega(A) \in \mathcal{S}(\mathcal{H})$ if

$$
A \in \mathbf{B}_1 := \{A \mid A \in \mathcal{L}_{2sym}(\mathcal{H}),\ \|A\|_{HS} < 1\} \subset \mathbf{D}_1 \subset \mathcal{L}_{2sym}(\mathcal{H})
$$

and $A \in \mathbf{B}_1 \to \exp\Omega(A) \in \mathcal{S}(\mathcal{H})$ is analytic. The ball $\mathbf{B}_1$ is an open subset of the convex open set $\mathbf{D}_1 \subset \mathcal{L}_{2sym}(\mathcal{H})$.

For the subsequent calculations we use a diagonalization of the symmetric tensors $\Omega(A)$.

Lemma 8 *Let* $A \in \mathcal{L}_{2sym}(\mathcal{H})$ *then* $\Omega(A)$ *has a representation*

$$
\Omega(A) = \frac{1}{2}\sum_{\mu=0}^{\infty}\alpha_\mu\, f_\mu \vee f_\mu, \tag{78}
$$

where $\{f_\mu\}$ *is a set of orthonormal vectors in* $\mathcal{H}$ *and the* α_μ *are complex numbers such that the series* $\sum_\mu\left|\alpha_\mu\right|^2 = \left\|\Omega(A)\right\|^2 = \frac{1}{2}\left\|A\right\|_{HS}^2$ *converges. The vectors* $\{f_\mu\}$ *can be chosen such that* $\alpha_\mu \ge 0$.

Proof. See Lemma 2 of [17]. The corresponding representation of the operator A is

$$Af = \sum_{\mu=0}^{\infty} \alpha_\mu f_\mu \langle f_\mu \mid f \rangle = \sum_{\mu=0}^{\infty} \alpha_\mu f_\mu \left(f_\mu^* \mid f \right). \tag{79}$$

A direct proof of the representation (79) follows from Sect. 2.2 of [8]. □

In the main part of this subsection we calculate the function

$$\Phi(\bar{A}, B, f) := (\exp \Omega(A) \mid \exp \Omega(B) \vee \exp f) = \left\langle \exp \Omega(\bar{A}) \mid \exp \Omega(B) \vee \exp f \right\rangle \tag{80}$$

for A, $B \in \mathbf{B}_1$ and $f \in \mathcal{H}$. This function is antianalytic in A (analytic in $\bar{A}$) and analytic in B; it is uniquely determined by its values on the diagonal $B = A$. The tensor $\Omega(A)$ has the representation (78) with the additional constraint $\|A\|_{HS} = \sum_\mu |\alpha_\mu|^2 = \frac{1}{2} \|A\|_{HS} < 1/2$. That yields the product representations $\exp \Omega(A) = \prod_\mu \exp \left(\frac{1}{2} \alpha_\mu e_\mu \vee e_\mu \right)$ and

$$\begin{aligned}
\Phi(\bar{A}, A, f) &= \prod_\mu \varphi(\overline{\alpha_\mu}, \alpha_\mu, \gamma_\mu) \quad \text{with} \qquad (81)\\
\varphi(\overline{\alpha_\mu}, \alpha_\mu, \gamma_\mu) &= \sum_{k,m,n=0}^{\infty} \frac{1}{k!m!(2n)!} \left(\left(\frac{\alpha_\mu}{2} e_\mu \vee e_\mu \right)^k \mid \left(\frac{\alpha_\mu}{2} e_\mu \vee e_\mu \right)^m \vee (\gamma_\mu e_\mu)^{2n} \right) \\
\gamma_\mu &= (e_\mu \mid f).
\end{aligned}$$

The inner product vanishes unless $k = m + n$. The remaining sum

$$\varphi(\overline{\alpha}, \alpha, \gamma) = \sum_{m,n=0}^{\infty} \frac{2^{-2m-n} \overline{\alpha}^{m+n} \alpha^m \gamma^{2n}}{(m+n)!m!(2n)!} (2m+2n)!$$

can be evaluated using the identity

$$\sum_{m=0}^{\infty} \frac{(2m+2n)!}{(m+n)!m!} z^m = \frac{(2n)!}{n!} \sum_k \left(\frac{2n+1}{2} \right)_k \frac{(4z)^k}{k!} = \frac{(2n)!}{n!} (1-4z)^{-\frac{2n+1}{2}}$$

such that

$$\varphi(\overline{\alpha}, \alpha, \gamma) = \left(1 - |\alpha|^2\right)^{-\frac{1}{2}} \sum_n \frac{1}{n!} 2^{-n} (1-|\alpha|^2)^{-n} \overline{\alpha}^n \gamma^{2n} = \left(1 - |\alpha|^2\right)^{-\frac{1}{2}} \exp \left[\frac{1}{2} (1 - |\alpha|^2)^{-1} \overline{\alpha} \gamma^2 \right].$$

Then

$$\Phi(\bar{A}, A, f) = \det \left(I - \bar{A} A \right)^{-\frac{1}{2}} \exp \frac{1}{2} \left\langle f \mid (I - \bar{A}A)^{-1} \bar{A} f \right\rangle$$

follows. The function (80) can therefore be written as

$$(\exp \Omega(A) \mid \exp \Omega(B) \vee \exp f) = \det \left(I - A^+ B \right)^{-\frac{1}{2}} \exp \frac{1}{2} \left\langle f \mid (I - A^+ B)^{-1} A^+ f \right\rangle. \tag{82}$$

This identity yields the inner product (12)

$$(\exp \Omega(A) \mid \exp \Omega(B)) = \left(\det{}_{\mathcal{H}} \left(I - A^+ B \right) \right)^{-\frac{1}{2}} = \left(\det{}_{\mathcal{H}} \left(I - B A^+ \right) \right)^{-\frac{1}{2}}, \tag{83}$$

so far derived for A, $B \in \mathbf{B}_1$. But the calculations of (81) are well defined for $A \in \mathbf{D}_1$, and the right hand side of (83) is antianalytic for $A \in \mathbf{D}_1$ and analytic for $B \in \mathbf{D}_1$. Hence the mapping $A \in \mathbf{B}_1 \to \exp \Omega(A) \in \mathcal{S}(\mathcal{H})$ can be analytically continued to $A \in \mathbf{D}_1$, and the inner product of two of these vectors is given by (12).

We can now apply the arguments of the Appendix A.1 to define the ultracoherent vector $\Phi(A, f) = \exp \Omega(A) \vee \exp f$ for $A \in \mathbf{D}_1$ and $f \in \mathcal{H}$. Using (72) we verify the identity

$$(\exp(h+f) \mid \exp \Phi(B,g)) = (\exp h \mid \exp \Phi(B, Bf^* + g)\, \mathrm{e}^{\frac{1}{2}\langle f^* | Bf^* \rangle + \langle f^* | g \rangle}$$

for $B \in \mathbf{D}_1$ and $f, g, h \in \mathcal{H}$. Since $\mathcal{S}_{coh}(\mathcal{H})$ is dense in $\mathcal{S}^{(\alpha)}(\mathcal{H})$, $\alpha \in (0,1)$, this identity and Lemma 7 imply

$$(H \vee \exp f \mid \exp \Phi(B,g)) = (H \mid \exp \Phi(B, Bf^* + g)\, \mathrm{e}^{\frac{1}{2}\langle f^* | Bf^* \rangle + \langle f^* | g \rangle} \tag{84}$$

for all $H \in \mathcal{S}^{(\alpha)}(\mathcal{H})$, $\alpha \in (0,1)$. Given $A \in \mathbf{D}_1$ the vector $\exp \Omega(A)$ is an element of $\mathcal{S}^{(\alpha)}(\mathcal{H})$ for some $\alpha \in (0,1)$. Choosing $H = \exp \Omega(A)$ we obtain from (82) and (84)

$$\begin{aligned} &(\exp(\Omega(A) + f) \mid \exp(\Omega(B) + g)) = \det(I - A^+ B)^{-\frac{1}{2}} \\ &\times \mathrm{e}^{\frac{1}{2}\langle f^* | Bf^* \rangle + \langle f^* | g \rangle} \exp \tfrac{1}{2} \left\langle (Bf^* + g) \mid (I - A^+ B)^{-\frac{1}{2}} A^+ (Bf^* + g) \right\rangle \\ &= \exp\left(\tfrac{1}{2} \langle f^* \mid Cf^* \rangle + \langle f^* \mid (I - BA)^{-1} g \rangle + \tfrac{1}{2} \langle g \mid Dg \rangle \right) \end{aligned} \tag{85}$$

with the operators (15). Hence we have given another proof of (14).

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